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Kac Polymers*

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We show how a polymer in two dimensions with a self-repelling interaction of Kac type exhibits a diffusive–ballistic transition if considered on the appropriate scale.

KEY WORDS: Polymers; Kac potentials; phase transition.

1. INTRODUCTION

In the last two decades a considerable effort has been spent to find a mathematical description of the physical objects known as *polymers*. According to ref. 1: "a polymer is a long chain of molecules with two characteristic properties: an irregular shape and a certain stiffness". The natural mathematical object to describe irregular long chains is the random walk; since the precursory work by Brydges and Spencer,(2) the (weak) interaction added to the standard probability distribution of the random walk to take into account the stiffness of the polymer has been a self-repelling interaction. The simplest way to describe this model is to consider walks ω_t in $\mathbb{Z}^d$ starting at the origin and consisting of $|\omega| = N$ nearest neighbor steps, and to assign to each walk a weight proportional to

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$$p_N(\omega) = \exp\left[-\beta \sum_{0 \leqslant s < t \leqslant N} \delta(\omega_s - \omega_t) \right]. \tag{1.1}$$

A natural quantity to study is the mean square end-to-end distance of ω defined as:

$$R_N(\beta) = \frac{\sum_{|\omega|=N} \omega_N^2 p_N(\omega)}{\sum_{|\omega|=N} p_N(\omega)}. \tag{1.2}$$

A polymer is classified by the value of the coefficient α such that

$$\lim_{N \to \infty} \frac{R_N(\beta)}{N^\alpha} = D, \tag{1.3}$$

with D a positive coefficient; a polymer is said to be *diffusive* if $\alpha = 1$, *super-diffusive* if $1 < \alpha < 2$ and *ballistic* if $\alpha = 2$. The available rigorous results and the numerical simulations show that, although D depends on the coefficient β appearing in Eq. (1.1), the parameter α, and therefore the long distance behavior of the polymers, are β-independent.

Due to the high non locality of the interaction appearing in Eq. (1.1), the rigorous results about the behavior of Eq. (1.2) are quite scarce. In ref. 2 it is proved that for $d \geqslant 5$ the self-repelling polymers are diffusive, in ref. 3 it is shown the ballistic behavior for the one-dimensional case. Such results, together with a more detailed description of the behavior of the polymer, are contained in many other works, see e.g. ref. 4 and reference therein.

The ingenious technique that allows to prove the diffusive results is the *lace expansion*, which is a way to show by induction that the properties of the correlation functions of self-repelling polymers are the same as those of the free ones. There are many ways to implement such inductive hypothesis, and the recent versions of the lace expansion are very elegant, see e.g. refs. 3, 5. The same technique allows to show the ballistic behavior in $d = 1$.[5] The lace expansion, however, seems to be useless in the intermediate cases, i.e. for $d = 2, 3, 4$, where a super-diffusive but sub-ballistic behavior is expected. The only result in this direction is ref. 6, where the logarithmic corrections to the diffusion coefficient in $d = 4$ have been studied using a field theoretic approach in a continuous space.

Another interesting class of results concerns the so called *forgetful* or *elastic* polymers, i.e. when the self repulsion is weaker if the polymers intersect after a long time.[3] The weight associated to the walk ω is modified in the following way:

$$p_N^f(\omega) = \exp\left[-\beta \sum_{0 \leqslant s < t \leqslant N} f(|t-s|)\delta(\omega_s - \omega_t) \right], \tag{1.4}$$

where $f(x) > 0$ has a suitable decay for large x. Also in this case, however, the basic technique is the lace expansion, and the available rigorous results cover only the diffusive regime.

One may wonder why a genuine elastic interaction, i.e. a bending contribution in the energy associated to the polymers, is not considered in literature. This would be a way to take into account very naturally the stiffness of the polymer. A local interaction of this form, indeed, does not affect the diffusive behavior of the free random walk.

This can be easily seen in the following way: let us describe the walk ω in terms of its unit increments. To each ω it can be associated a set of unit vectors $\sigma_i, i = 1, \ldots, N$, taking values in $\{\pm e_1, \ldots, \pm e_d\}$, where $e_1, \ldots, e_d$ are the generators of the lattice $\mathbb{Z}^d$ and represent the ith step of the walk. Clearly $\omega_t = \sum_{i=1}^t \sigma_i$. A simple way to assign a weight to each ω preventing the bending of the polymer would be:

$$p_N^{\mathrm{b}}(\omega) = p_N^{\mathrm{b}}(\sigma) = \exp\left[\beta \sum_{i=1}^N \sigma_i \cdot \sigma_{i+1} \right], \tag{1.5}$$

and the mean square end-to-end distance of ω is defined in this context as:

$$R_N(\beta) = \sum_{1 \leqslant i,j \leqslant N} \frac{\sum_\sigma \sigma_i \cdot \sigma_j \, p_N^{\mathrm{b}}(\sigma)}{\sum_\sigma p_N^{\mathrm{b}}(\sigma)} = \sum_{1 \leqslant i,j \leqslant N} \langle \sigma_i \cdot \sigma_j \rangle. \tag{1.6}$$

The measure obtained in this way can be interpreted as the Gibbs measure for a bounded spin model in one dimension with finite range interactions, and it is well known[7,8] that these models do not exhibit phase transitions for any temperature and their spin–spin correlations $\langle \sigma_i \cdot \sigma_j \rangle$ decay exponentially. This implies from Eq. (1.6) simple diffusion. A bending energy different from the simplest choice in Eq. (1.5), but with finite range, would give always diffusive polymers.

In recent times, however, some interesting results about one-dimensional spin models have been proved. In particular, for the class of interactions known as *Kac potentials*, the existence of non-zero magnetization domains on a scale much larger than the range of the potential has been proved in ref. 9. This behavior suggests the possibility that a polymer with

finite range self interaction can exhibit phase transitions as a *finite size effect*.

In this work we propose a model of polymers (Kac polymers) defined in a two-dimensional space which can be rewritten in terms of a one-dimensional bounded spin system very similar to the one described in ref. 9. With the same techniques used there, we are able to prove a diffusive–ballistic phase transition for this class of polymers, when the infinite volume limit is realized in a suitable sense or, which is the same, if the polymer is seen on an appropriate scale.

The result is stated for polymers on a two-dimensional lattice space since in this case, by a suitable change of variables, the Hamiltonian can be disentangled as the sum of two Ising Hamiltonians with Kac interaction. The generalization to higher spatial dimension does not present, in principle, any obstacle but is technically much more involved.

Hence the above diffusive–ballistic phase transition for Kac polymers does not depend on the spatial dimension d. This is a relevant difference w.r.t. the self-repelling models above mentioned. This is due to the local character of the interaction and to the fact that the model we introduce is intrinsically one-dimensional.

To fix the ideas a possible model (that we will slightly modify in the next section) is the following one. Let us consider a walk $\omega = \{\omega_i\}_{i=0,1,2,\dots,N}$, on a $d = 2$-dimensional lattice; we define the following weight for the walk:

$$p_N^{\mathrm{K}}(\omega) = \exp\left[\beta\gamma^3 \sum_{\substack{0 \leqslant i < j \leqslant N \\ j-i \leqslant \gamma^{-1}}} (\omega_i - \omega_j)^2\right], \tag{1.7}$$

where $\beta > 0$ and $\gamma > 0$ are suitable parameters. Take, for simplicity, γ such that γ^{-1} is integer.

This weight tends to prevent the bending of the polymer. The interaction is always finite range, but when γ is small its range tends to increase, as usual in Kac potential. We will give the exact statement of the main theorem of this paper in the following section, however our result can be summarized in the following way: there exists $\beta_c > 0$ (which can be exactly computed, see below) such that, given $\epsilon > 0$, there exists a constant $c > 0$ (depending on ϵ) for which the following holds. If we choose β such that $|\beta - \beta_c| \geqslant \epsilon$ and $N = e^{c\gamma^{-1}}$, then the polymer described by Eq. (1.7) is ballistic for $\beta > \beta_c$, while it is at most diffusive for $\beta < \beta_c$.

In the limit $\gamma \to 0$, rescaling the space with a factor γ^{-1}, one formally obtains a continuous walk, with a bending interaction on scale 1. The interaction is such that the repulsion between finite small contiguous segments is independent on γ (this is the reason of the factor γ^3 in front of the sum appearing in the Hamiltonian).

One might also characterize the transition by considering polymers of infinite length, by means of the (unique) DLR infinite volume measure relative to the interaction in Eq. (1.7). More precisely, from the results of ref. 9, we could prove that, for $\beta > \beta_c$ and $\gamma \to 0$, there exists a definite length scale of order $e^{\phi\gamma^{-1}}$ with ϕ a suitable constant, such that on this length scale the typical configurations of the polymer are polygons of random Poisson length. In particular this implies that polymers of finite length N much larger than $e^{\phi\gamma^{-1}}$ behave diffusively. Conversely, if $\beta < \beta_c$, we could prove that on a scale slightly larger than γ^{-1}, the polymer behaves as a Brownian path as $\gamma \to 0$.

There are several natural ways to modify this model. One possibility is to study with the same technique a random walk on the continuous space with steps of length 1 and random (continuous) direction. This corresponds to the study of one-dimensional planar rotator or classical Heisenberg models with Kac interaction. Some partial results on these models can be found in ref. 10, which suggest that the rigidity of the polymer is in this case on scale $c\gamma^{-2}$ instead of the much larger $e^{c\gamma^{-1}}$. Another possible model is to replace the Kac potential with a true long range interaction, e.g. decaying as $1/r^2$. This choice has the advantage to produce a genuine ballistic–diffusive phase transition, as it could be proved using the results of ref. 11, but this seems to us less reasonable from a physical point of view for a polymer because its elastic stiffness is intrinsically local.

2. NOTATION AND RESULTS

We denote by $\mathbb{Z}^2$ the two-dimensional cubic lattice. We will denote by e_1 and e_2 the generators of the lattice $\mathbb{Z}^2$, namely $e_1 = (1, 0)$ and $e_2 = (0, 1)$. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are two elements of $\mathbb{Z}^2$, then $x + y = (x_1 + y_1, x_2 + y_2)$; moreover $x \cdot y$ denotes the usual scalar product $x \cdot y = x_1 y_1 + x_2 y_2$ and $|x - y|$ the Euclidean distance $|x - y| = \sqrt{(x - y) \cdot (x - y)}$. Finally, if $x \in \mathbb{Z}^2$ then $x^2 \doteq x \cdot x$.

Given a pair $\{i, j\} \subset \mathbb{N}$ with $i < j$, we denote by $[i, j]$ the set $[i, j] = \{i, i + 1, \ldots, j - 1, j\}$ and define $\mathcal{N} \doteq [1, N] = \{1, 2, \ldots, N\}$ and $\mathcal{N}_0 \doteq [0, N] = \{0, 1, 2, \ldots, N\}$.

An *N-step random walk* from the origin is a function $\omega: \mathcal{N}_0 \to \mathbb{Z}^2 : i \mapsto \omega_i$ such that $\omega_0 = 0$ and $|\omega_i - \omega_{i-1}| = 1$ for all $i = 1, 2, \ldots, N$. The value

$\omega_i \in \mathbb{Z}^2$ is the *position of the walk ω after i steps*. We denote by Ω_N the set of all N-step random walks in $\mathbb{Z}^2$ starting at 0. Given a random walk $\omega \in \Omega_N$, the function $\sigma_{\mathcal{N}}: \mathcal{N} \to \{e_1; -e_1; e_2; -e_2\}: i \mapsto \sigma_i \doteq \omega_i - \omega_{i-1}$ is called *the spin configuration associated to ω*. Note that the value $\sigma_i = \omega_i - \omega_{i-1}$ represents the ith step of the walk ω. We denote by Σ_N the set of all possible spin configurations associated to some ω. Clearly the correspondence $\omega \leftrightarrow \sigma_{\mathcal{N}}$ is by construction 1–1 and onto. Given a set $\Lambda \subset \mathcal{N}$, we will denote by σ_Λ a generic spin configuration in Λ, namely σ_Λ is a function $\sigma_\Lambda: \Lambda \to \{e_1, -e_1, e_2, -e_2\}$, and we denote by Σ_Λ the set of all spin configuration in Λ.

We now assign a Kac type self-interaction to each walk ω. A Kac interaction is defined as follows. Let us take a real, non-negative, even and differentiable function $t \mapsto \Phi(t)$, $t \in \mathbb{R}$, compactly supported in $[-1, 1]$ and positive in $(-1, 1)$. The Kac potential Φ_γ, $\gamma \in (0, 1]$ induced by Φ is defined as:

$$\Phi_\gamma(t) = \gamma \Phi(\gamma t). \tag{2.1}$$

Given now an N-step random walk from the origin $\omega \in \Omega_N$, the *energy* $E_N(\omega)$ of ω is defined as:

$$E_N(\omega) \doteq -\gamma^2 \sum_{0 \leqslant i < j \leqslant N} \Phi_\gamma(j - i)\,(\omega_j - \omega_i)^2. \tag{2.2}$$

We will assume hereafter $\gamma \in \{2^{-n}; n \in \mathbb{N}\}$ (so γ^{-1} is an integer) and that also γN is an integer. We remark that this assumption does not hide a technical difficulty: we want just to avoid the use of heavier notation involving integer parts.

Moreover, we are interested in the behavior of the random walks when its length N is much larger than the interaction range γ^{-1} (in fact exponentially large in γ^{-1}).

According to the Gibbs prescription, we can now define the probability of a given walk $\omega \in \Omega_N$ at a fixed inverse temperature $\beta > 0$ as:

$$p_{\beta,\gamma}(\omega) \doteq \frac{e^{-\beta E_N(\omega)}}{\Xi_N(\beta, \gamma)}, \tag{2.3}$$

where $\Xi_N(\beta, \gamma)$ is the partition function, i.e. the normalization factor given by

$$\Xi_N(\beta, \gamma) = \sum_{\omega \in \Omega_N} e^{-\beta E_N(\omega)}. \tag{2.4}$$

We are interested in the *mean quadratic end-to-end distance*:

$$R_{N,\gamma}(\beta) \doteq \sum_{\omega \in \Omega_N} p_{\beta,\gamma}(\omega)\omega_N^2 \tag{2.5}$$

of a random walk in the ensemble described by the probability measure (2.3). Our main result is the content of the following theorem.

Theorem 2.1. There exist $\beta_c > 0$ such that the following holds. If $\beta > \beta_c$, let $N = e^{a\gamma^{-1}}$; then there are $a_\beta > 0$ and $D_\beta > 0$ such that:

$$\lim_{\gamma \downarrow 0} \frac{R_{N,\gamma}(\beta)}{N^2} = D_\beta \quad \forall a \in (0, a_\beta). \tag{2.6}$$

If $\beta < \beta_c$ there are $\gamma_\beta \in (0, 1)$ and $C_\beta > 0$ such that:

$$\frac{R_{N,\gamma}(\beta)}{N} \leqslant \frac{C_\beta}{\gamma} \quad \forall N \geqslant 1, \ \ \forall \gamma \in (0, \gamma_\beta). \tag{2.7}$$

The proof of Theorem 2.1 will be given at the end of the paper. In the following sections we will present a formulation of the model in terms of a one-dimensional spin system and we list a series of results on this spin model. Note that as far as the diffusive regime is concerned we just give a bound, stating that the polymer can be at most diffusive on the scale of the interaction. This guarantees the existence of the two regimes.

3. SPIN SYSTEM REFORMULATION

We write here the model in terms of the "spin variables" $\sigma_i = \omega_i - \omega_{i-1}$ defined in the previous section. We have:

$$\begin{aligned}
\gamma^2 \sum_{0 \leqslant i < j \leqslant N} \Phi_\gamma(j-i)\,(\omega_j - \omega_i)^2 &= \gamma^2 \sum_{0 \leqslant i < j \leqslant N} \Phi_\gamma(j-i) \left(\sum_{k=i+1}^{j} \sigma_k \right)^2 \\
&= \gamma^2 \sum_{0 \leqslant i < j \leqslant N} \Phi_\gamma(j-i) \left[2 \sum_{i < k < l \leqslant j} \sigma_k \cdot \sigma_l + (j-i-1) \right] \\
&= 2\gamma^2 \sum_{0 \leqslant i < j \leqslant N} \Phi_\gamma(j-i) \sum_{i < k < l \leqslant j} \sigma_k \cdot \sigma_l + C_N,
\end{aligned} \tag{3.1}$$

where

$$C_N = N\gamma^2 \sum_{s=1}^{N} \Phi_\gamma(s)(s-1) - \gamma^2 \sum_{s=1}^{N} \Phi_\gamma(s)s(s-1),$$

so that $C_N = \gamma N O(1)$. We then write:

$$\begin{aligned}
&2\gamma^2 \sum_{0\leqslant i<j\leqslant N} \Phi_\gamma(j-i) \sum_{i<k<l\leqslant j} \sigma_k\cdot\sigma_l \\
&\quad = 2\gamma^2 \sum_{1\leqslant k<l\leqslant N} \sigma_k\cdot\sigma_l \sum_{\substack{\{i,j\}\in\{0,1,2,\ldots,N\}\\ [i,j]\supseteq[k-1,l]}} \Phi_\gamma(j-i) \\
&\quad = 2\gamma^2 \sum_{1\leqslant k<l\leqslant N} \sigma_k\cdot\sigma_l \sum_{s\geqslant 1} \sum_{\substack{\{i,j\}\in\{0,1,2,\ldots,N\}\\ [i,j]\supseteq[k-1,l],\ j-i=s+l-k}} \Phi_\gamma(j-i) \\
&\quad = \sum_{1\leqslant k<l\leqslant N} V_\gamma(k,l)\,\sigma_k\cdot\sigma_l,
\end{aligned}$$

where

$$V_\gamma(k,l) \doteq 2\gamma^2 \sum_{s\geqslant 1} \Phi_\gamma(s+l-k) F_{k,l}(s) \tag{3.2}$$

and

$$F_{k,l}(s) = \begin{cases} s & \text{if } l+s\leqslant N \text{ and } s<k, \\ N-l & \text{if } l+s>N \text{ and } s<k, \\ k & \text{if } l+s\leqslant N \text{ and } s\geqslant k, \\ \min\{k, N-l\} & \text{if } l+s>N \text{ and } s\geqslant k. \end{cases} \tag{3.3}$$

We have thus written $E_N(\omega)$ as:

$$E_N(\omega) = E^{(0)}_{\mathcal{N}}(\sigma_{\mathcal{N}}) + C_N, \tag{3.4}$$

where

$$E^{(0)}_{\mathcal{N}}(\sigma_{\mathcal{N}}) = -\sum_{1\leqslant k<l\leqslant N} V_\gamma(k,l)\,\sigma_k\cdot\sigma_l. \tag{3.5}$$

Formulas (3.4) and (3.5) above show that the energy $E_N(\omega)$ describing the self interaction of a two-dimensional random walk $\omega \in \Omega_N$ can be reinterpreted as the energy $E^{(0)}_{\mathcal{N}}(\sigma_{\mathcal{N}})$ of a *one-dimensional* spin system confined in the "volume" $\mathcal{N}$ with *free boundary conditions*. Namely, $E^{(0)}_{\gamma}(\sigma)$ represents the energy of the spin configuration $\sigma \in \Sigma_N$ uniquely associated to ω. The spins in this one-dimensional system are two-dimensional vectors assuming the four possible values $\pm e_1, \pm e_2$ and they interact via an Ising ferromagnetic interaction with pair potential $V_\gamma(k,l)$ defined in Eq. (3.2). Finally, the partition function (2.4) can now be written, as:

$$\Xi_N(\beta,\gamma) = e^{\beta C_N}\, \Xi^{(0)}_{\mathcal{N}}(\beta,\gamma),$$

where

$$\Xi^{(0)}_{\mathcal{N}}(\beta,\gamma) = \sum_{\sigma_{\mathcal{N}} \in \Sigma_N} e^{-\beta E^{(0)}_{\mathcal{N}}(\sigma_{\mathcal{N}})} \tag{3.6}$$

is the partition function of the one-dimensional spin system above.

By the hypothesis on Φ_γ, it is immediate to check that $V_\gamma(k,l)$ has finite range equal to γ^{-1}, namely:

$$V_\gamma(k,l) = 0 \quad \text{whenever } l-k \geqslant \gamma^{-1}. \tag{3.7}$$

We would also like $V_\gamma(k,l)$ to be *translational invariant*, namely we would like $V_\gamma(k,l)$ to depend only on the difference $l-k$. This in general is not true because the function $F_{k,l}(s)$ defined in Eq. (3.3) is not translational invariant. Actually it is easy to see that such loss of translational invariance occurs only when k and l vary in a very small set near the borders of $\mathcal{N}$. To make things more precise let us define, for $t \in \mathbb{R}$,

$$J_\gamma(t) = 2\gamma^2 \sum_{s \geqslant 1} s\, \Phi_\gamma(s + |t|) \tag{3.8}$$

and consider the subsets Δ_1 and Δ_N of $\mathcal{N}$ given by $\Delta_1 = \{k \in \mathcal{N} : k \leqslant \gamma^{-1}\}$ and $\Delta_N = \{l \in \mathcal{N} : N - \gamma^{-1} < l \leqslant N\}$. Then, due to the finite range property (3.7), one can easily check that:

$$V_\gamma(k,l) = J_\gamma(l-k) + \epsilon_\gamma(k,l)$$

with

$$\epsilon_\gamma(k,l)=\begin{cases}0 & \text{if } \{k,l\}\not\subset(\Delta_1\cup\Delta_N),\\ 2\gamma^2\sum_{s=1}^{\infty}[F_{kl}(s)-s)]\Phi_\gamma(s+l-k) & \text{if } \{k,l\}\subset(\Delta_1\cup\Delta_N).\end{cases}$$

In other words, the pair potential $V_\gamma(k,l)$ is indeed a function $J_\gamma(l-k)$ of the difference $l-k$ if k,l are not both contained in one of the two extremal sets Δ_1 and Δ_N. When vice versa k and l both belong either to one of the two extremal sets, then $V_\gamma(k,l)$ is given by $J_\gamma(l-k)$ plus the non-translational invariant "boundary" correction $\epsilon_\gamma(k,l)$.

The function $J_\gamma(t)$ defined in Eq. (3.8) behaves as a Kac potential. Indeed:

$$J_\gamma(t)=\gamma J^{(\gamma)}(\gamma t) \tag{3.9}$$

where

$$J^{(\gamma)}(t)=\gamma\sum_{s=1}^{\infty}(\gamma s)\,\Phi(\gamma s+|t|)$$

is such that

$$\lim_{\gamma\to 0}J^{(\gamma)}(t)\doteq J^{(0)}(t)=\int_0^\infty dx\,x\,\Phi(x+|t|). \tag{3.10}$$

It is now clear the reason of the presence of the scaling factor γ^2 in the Hamiltonian (2.2): with this choice the function $J_\gamma(t)$ has the correct scaling (3.9) (the function $J^{(\gamma)}(t)$ being almost "scale invariant").

By definition (3.8) and the properties of Φ, it follows that $J(t)$ is an even, non-negative, differentiable function, compactly supported in $[-1,1]$ and positive in $(-1,1)$.

Let now $\mathcal{N}^{\mathrm{b}}\doteq\mathcal{N}\setminus(\Delta_1\cup\Delta_N)$, then any configuration $\sigma_{\mathcal{N}}$ on $\mathcal{N}$ can be seen as the union of a configuration σ_{Δ_1} on Δ_1, σ_{Δ_N} on Δ_N, and $\sigma_{\mathcal{N}^{\mathrm{b}}}$ on $\mathcal{N}^{\mathrm{b}}$. Therefore, using Eq. (3.7), one can easily check that

$$E^{(0)}_{\mathcal{N}}(\sigma_{\mathcal{N}})=E^{(0)}_{\Delta_1}(\sigma_{\Delta_1})+E^{(0)}_{\Delta_N}(\sigma_{\Delta_N})+H^{\sigma_{\Delta_1\cup\Delta_N}}_{\mathcal{N}^{\mathrm{b}}}(\sigma_{\mathcal{N}^{\mathrm{b}}}),$$

where $\sigma_{\Delta_1\cup\Delta_N}$ is the spin configuration on $\Delta_1\cup\Delta_N$ which agrees with σ_{Δ_1} on Δ_1 and with σ_{Δ_N} on Δ_N, and $H^{\sigma_{\Delta_1\cup\Delta_N}}_{\mathcal{N}^{\mathrm{b}}}(\sigma_{\mathcal{N}^{\mathrm{b}}})$ is a spin Hamiltonian

defined in the volume $\mathcal{N}^{\mathfrak{b}}$ with boundary condition $\sigma_{\Delta_1 \cup \Delta_N}$ through the translational invariant Kac potential J_γ:

$$H_{\mathcal{N}^{\mathfrak{b}}}^{\sigma_{\Delta_1 \cup \Delta_N}}(\sigma_{\mathcal{N}^{\mathfrak{b}}}) = - \sum_{\substack{\{i<j\} \\ i \in \Delta_1 \cup \mathcal{N}^{\mathfrak{b}},\, j \in \mathcal{N}^{\mathfrak{b}} \cup \Delta_N}} J_\gamma(j-i)\, \sigma_i \cdot \sigma_j. \tag{3.11}$$

The partition function (3.6) can be rewritten as

$$\Xi_{\mathcal{N}}^{(0)}(\beta, \gamma) = \sum_{\substack{\sigma_{\Delta_1} \in \Sigma_{\Delta_1} \\ \sigma_{\Delta_N} \in \Sigma_{\Delta_N}}} e^{-\beta E_{\Delta_1}^{(0)}(\sigma_{\Delta_1}) + E_{\Delta_N}^{(0)}(\sigma_{\Delta_N})}\, Z_{\mathcal{N}^{\mathfrak{b}}}^{\sigma_{\Delta_1 \cup \Delta_N}}(\beta, \gamma), \tag{3.12}$$

where

$$Z_{\mathcal{N}^{\mathfrak{b}}}^{\sigma_{\Delta_1 \cup \Delta_N}}(\beta, \gamma) = \sum_{\sigma_{\mathcal{N}^{\mathfrak{b}}} \in \Sigma_{\mathcal{N}^{\mathfrak{b}}}} e^{-\beta H_{\mathcal{N}^{\mathfrak{b}}}^{\sigma_{\Delta_1 \cup \Delta_N}}(\sigma_{\mathcal{N}^{\mathfrak{b}}})}.$$

Consider now the finite volume Gibbs measure on $\Sigma_{\mathcal{N}^{\mathfrak{b}}}$,

$$\nu_{\beta,\gamma}^{\sigma_{\Delta_1 \cup \Delta_N}}(\sigma_{\mathcal{N}^{\mathfrak{b}}}) = \frac{e^{-\beta H_{\mathcal{N}^{\mathfrak{b}}}^{\sigma_{\Delta_1 \cup \Delta_N}}(\sigma_{\mathcal{N}^{\mathfrak{b}}})}}{Z_{\mathcal{N}^{\mathfrak{b}}}^{\sigma_{\Delta_1 \cup \Delta_N}}(\beta, \gamma)}. \tag{3.13}$$

We now show that under a suitable change of variables, the measure (3.13) is mapped into the product of two independent Ising models. We introduce the two orthogonal vectors in $\mathbb{R}^2$,

$$f_1 = \frac{e_1 + e_2}{2}, \qquad f_2 = \frac{e_1 - e_2}{2},$$

so that $e_1 = f_1 + f_2$ and $e_2 = f_1 - f_2$. Then, to each spin configuration $\sigma \in \Sigma_N$ it corresponds a unique pair $(\tau^{(1)}, \tau^{(2)})$, where

$$\tau^{(r)} = (\tau_1^{(r)}, \ldots, \tau_N^{(r)}) \in M_N \doteq \{-1, 1\}^{\mathcal{N}}, \quad r = 1, 2,$$

such that $\sigma_i = \tau_i^{(1)} f_1 + \tau_i^{(2)} f_2$ for all $i \in \mathcal{N}$. Under the map $\sigma \to \tau$ the measure (3.13) is mapped into a product measure. Indeed:

$$H_{\mathcal{N}^{\mathfrak{b}}}^{\sigma_{\Delta_1 \cup \Delta_N}}(\sigma_{\mathcal{N}^{\mathfrak{b}}}) = H_{\mathcal{N}^{\mathfrak{b}}}^{\tau_{\Delta_1 \cup \Delta_N}^{(1)}}(\tau_{\mathcal{N}^{\mathfrak{b}}}^{(1)}) + H_{\mathcal{N}^{\mathfrak{b}}}^{\tau_{\Delta_1 \cup \Delta_N}^{(2)}}(\tau_{\mathcal{N}^{\mathfrak{b}}}^{(2)}),$$

where, for $r=1,2$,

$$H_{\mathcal{N}^b}^{\tau^{(r)}_{\Delta_1\cup\Delta_N}}(\tau^{(r)}_{\mathcal{N}^b})=-\frac{1}{2}\sum_{\substack{\{i<j\}\\ i\in\Delta_1\cup\mathcal{N}^b,\, j\in\mathcal{N}^b\cup\Delta_N}} J_\gamma(j-i)\,\tau_i^{(r)}\tau_j^{(r)}. \tag{3.14}$$

Then, for any real function $\sigma\to F(\sigma)$,

$$\nu_{\beta,\gamma}^{\sigma_{\Delta_1\cup\Delta_N}}(F)=\mu_{\beta,\gamma}^{\tau^{(1)}_{\Delta_1\cup\Delta_N}}\otimes\mu_{\beta,\gamma}^{\tau^{(2)}_{\Delta_1\cup\Delta_N}}(\tilde F), \tag{3.15}$$

where $\tilde F(\tau)\doteq F(\tau^{(1)}f_1+\tau^{(2)}f_2)$ and

$$\mu_{\beta,\gamma}^{\tau^{(r)}_{\Delta_1\cup\Delta_N}}(\tau_{\mathcal{N}^b})=\frac{e^{-\beta H_{\mathcal{N}^b}^{\tau^{(r)}_{\Delta_1\cup\Delta_N}}(\tau^{(r)}_{\mathcal{N}^b})}}{Z_{\mathcal{N}^b}^{\tau^{(r)}_{\Delta_1\cup\Delta_N}}(\beta,\gamma)},\quad r=1,2. \tag{3.16}$$

We have thus decomposed the system as the superposition of two independent one-dimensional Ising model with Kac interaction, whose typical behavior is studied in ref. 9. Theorem 2.1 is now an easy consequence of those results.

4. PROOF OF THEOREM 2.1

We shall prove Theorem 2.1 with

$$\beta_c=2\left[\int \mathrm{d}t\, J^{(0)}(t)\right]^{-1}, \tag{4.1}$$

i.e. the inverse mean field critical temperature for the one-dimensional Ising spin system with interaction $J_\gamma/2$ (recall definitions (3.10) and (3.14)).

Let us denote by $\mu_{\beta,\gamma}$ the (infinite volume) Gibbs measure on $\{-1,1\}^{\mathbb{Z}}$ with potential $J_\gamma/2$ at inverse temperature β. Clearly, if the event A is generated by the spin variables $\{\tau_i;\ i\in\mathcal{N}^b\}$ then:

$$\mu_{\beta,\gamma}(A|\tau^{(r)}_{\Delta_1\cup\Delta_N})=\mu_{\beta,\gamma}^{\tau^{(r)}_{\Delta_1\cup\Delta_N}}(A), \tag{4.2}$$

where $\mu_{\beta,\gamma}(\cdot|\tau^{(r)}_{\Delta_1\cup\Delta_N})$ denotes the conditional infinite volume measure.

By Theorem 5.1 in ref. 9 there exist two positive constants $\bar{a}$ and k such that, if A is any event generated by $\{\tau_i;\ i > \gamma^{-k}\}$ and symmetric under spin flip, then:

$$|\mu_{\beta,\gamma}(A|\tau_C) - \mu_{\beta,\gamma}(A|\tau'_C)| \leqslant e^{-\bar{a}\gamma^{-1}}, \tag{4.3}$$

for any $\tau_C, \tau'_C \in \{-1, 1\}^C$ and any interval $C = [-L, -1]$ in $\mathbb{Z}_{<0}$ with $L > \gamma^{-1}$.

We shall prove Eq. (2.6) with $a_\beta \leqslant \bar{a}/2$. Therefore we fix $\beta > \beta_c$, $a \in (0, \bar{a}/2)$, and $N = e^{a\gamma^{-1}}$. For k as above and γ small enough, let $\mathcal{N}_k^{\mathrm{b}}$ be the (non-empty) set $[\gamma^{-(k+1)}, N - \gamma^{-(k+1)}]$ and define

$$\omega^2_{\mathcal{N}_k^{\mathrm{b}}} = (\omega_{N-\gamma^{-(k+1)}} - \omega_{\gamma^{-(k+1)}})^2 = \sum_{i,j \in \mathcal{N}_k^{\mathrm{b}}} (\tau_i^{(1)}\tau_j^{(1)} + \tau_i^{(2)}\tau_j^{(2)}). \tag{4.4}$$

Since the function $\tau_i^{(r)}\tau_j^{(r)}$ is invariant under spin flip and it is generated by $\{\tau_i^{(r)};\ i \in \mathcal{N}^{\mathrm{b}}\}$, by Eqs. (3.15), (4.2), (4.3), and the total probability theorem,

$$\left| \nu_{\beta,\gamma}^{\sigma_{\Delta_1 \cup \Delta_N}} \left(\omega^2_{\mathcal{N}_k^{\mathrm{b}}} \right) - 2\,\mu_{\beta,\gamma} \left(\sum_{i \in \mathcal{N}_k^{\mathrm{b}}} \tau_i \right)^2 \right| \leqslant 2N^2 e^{-\bar{a}\gamma^{-1}} \leqslant 2. \tag{4.5}$$

Then, recalling Eqs. (2.5), (2.3), (2.4), (3.6), and (3.12), by (4.5) we get

$$\begin{aligned} \lim_{\gamma \downarrow 0} \frac{R_{N,\gamma}(\beta)}{N^2} &= \lim_{\gamma \downarrow 0} \frac{1}{|\mathcal{N}_k^{\mathrm{b}}|^2} \sum_{\omega \in \Omega_N} p_{\beta,\gamma}(\omega) \omega^2_{\mathcal{N}_k^{\mathrm{b}}} \\ &= \lim_{\gamma \downarrow 0} \frac{2}{|\mathcal{N}_k^{\mathrm{b}}|^2}\, \mu_{\beta,\gamma} \left(\sum_{i \in \mathcal{N}_k^{\mathrm{b}}} \tau_i \right)^2 . \end{aligned} \tag{4.6}$$

We now exploit the results of Theorems 2.3 and 2.8 in ref. 9. Given $\delta \in \{2^{-n};\ n \in \mathbb{N}\}$, we denote by I_ℓ, $\ell \in \mathbb{Z}$, the subset of $\mathbb{Z}$ defined as:

$$I_\ell = \{i \in \mathbb{Z} : \delta\gamma^{-1}(\ell - 1) < i \leqslant \delta\gamma^{-1}\ell\}. \tag{4.7}$$

In particular $\mathcal{N}_k^{\mathrm{b}}$ is partitioned into $L=\delta^{-1}\gamma|\mathcal{N}_k^{\mathrm{b}}|$ intervals $I_1,\dots,I_L$, each one containing exactly $\delta\gamma^{-1}$ sites of $\mathcal{N}_k^{\mathrm{b}}$. To each spin configuration $\tau\in\{-1,1\}^{\mathbb{Z}}$ we associate the mean magnetization in I_ℓ:

$$m_\ell=m_\ell(\tau)\doteq\gamma\delta^{-1}\sum_{i\in I_\ell}\tau_i. \tag{4.8}$$

Now let m_β be the positive solution of the mean field equation:

$$m_\beta=\tanh(\beta\beta_c^{-1}m_\beta).$$

By Theorems 2.3 and 2.8 in ref. 9, for any sufficiently small $c,\delta>0$ and $\zeta\in(0,m_\beta)$,

$$\lim_{\gamma\downarrow0}\mu_{\beta,\gamma}\left(\{\tau:|m_\ell-m_\beta|<\zeta\quad\forall\,|\ell|\leqslant\delta^{-1}\gamma e^{c\gamma^{-1}}\}\right)=\frac{1}{2}.$$

and the same holds with m_β replaced by $-m_\beta$. Hence, if a is small enough,

$$\mu_{\beta,\gamma}\left(\sum_{i\in\mathcal{N}_k^{\mathrm{b}}}\tau_i\right)^2=\frac{\delta^2}{\gamma^2}\mu_{\beta,\gamma}\left(\sum_{\ell,\ell'=1}^{L}m_\ell m_{\ell'}\right)=|\mathcal{N}_k^{\mathrm{b}}|^2\left[m_\beta^2+\varepsilon(\zeta,\gamma)\right],$$

with

$$\limsup_{\zeta\downarrow0}\limsup_{\gamma\downarrow0}\varepsilon(\zeta,\gamma)=0.$$

Then, choosing $a_\beta\in(0,\bar a/2)$ sufficiently small and recalling Eq. (4.6), we get Eq. (2.6) with $D_\beta=2m_\beta^2$.

Concerning the second part of Theorem 2.1, we first note that the estimate (2.7) is trivial if γN is uniformly bounded, whence we assume $N\geqslant10\gamma^{-1}$ in the sequel. By definition (4.1), if $\beta<\beta_c$ there exist $\gamma_\beta\in(0,1)$ and $r\in(0,1)$ such that:

$$\frac{\beta}{2}\sum_{j\neq i}J_\gamma(i-j)\leqslant r\quad\forall\,\gamma\in(0,\gamma_\beta). \tag{4.9}$$

This is the well known Dobrushin's uniqueness condition for the interaction $J_\gamma/2$ which implies in particular the following facts, see ref. 12. If Λ

is any interval in $\mathbb{Z}$ and $i \in \Lambda$, for any boundary conditions $\tau_{\Lambda^c}, \tau'_{\Lambda^c}$ there exists a coupling $Q^{\tau_{\Lambda^c}, \tau'_{\Lambda^c}}(\tau_\Lambda, \tau'_\Lambda)$ of the two probabilities $\mu^{\tau_{\Lambda^c}}_{\beta,\gamma}$ and $\mu^{\tau'_{\Lambda^c}}_{\beta,\gamma}$ such that:

$$D^{\tau_{\Lambda^c}, \tau'_{\Lambda^c}}_{\Lambda,i} \doteq \sum_{\tau_\Lambda, \tau'_\Lambda} Q^{\tau_{\Lambda^c}, \tau'_{\Lambda^c}}(\tau_\Lambda, \tau'_\Lambda) \, |\tau_i - \tau'_i| \leqslant 2 \sum_{n \geqslant 1} \sum_{j \notin \Lambda} \sum_{i_i, \ldots, i_{n-1}} r_{i,i_1} \cdots r_{i_{n-1},j},$$

where $r_{t,s} \doteq \beta J_\gamma(t-s)/2$. Since $r_{t,s} = 0$ for $|t-s| \geqslant \gamma^{-1}$, denoting by n_0 the integer part of $\gamma \operatorname{dist}(i, \Lambda^c)$, by Eq. (4.9) we have, for suitable $C_1, C_2 > 0$,

$$D^{\tau_{\Lambda^c}, \tau'_{\Lambda^c}}_{\Lambda,i} \leqslant 2 \sum_{n \geqslant n_0} r^n \leqslant C_1 e^{-C_2 \gamma \operatorname{dist}(i, \Lambda^c)} \quad \forall \gamma \in (0, \gamma_\beta). \tag{4.10}$$

Now, since $\mu_{\beta,\gamma}(\tau_i) = 0$,

$$|\mu^{\tau_{\Lambda^c}}_{\beta,\gamma}(\tau_i)| = |\mu^{\tau_{\Lambda^c}}_{\beta,\gamma}(\tau_i) - \mu_{\beta,\gamma}(\tau_i)| \leqslant \int d\mu_{\beta,\gamma}(\tau'_{\Lambda^c}) \, D^{\tau_{\Lambda^c}, \tau'_{\Lambda^c}}_{\Lambda,i}. \tag{4.11}$$

Moreover, if $i < j$ and $i, j \in \Lambda$, setting $I = [i, j-1] \subset \Lambda$ we also have:

$$\begin{aligned} &|\mu^{\tau_{\Lambda^c}}_{\beta,\gamma}(\tau_i \tau_j) - \mu^{\tau_{\Lambda^c}}_{\beta,\gamma}(\tau_i) \, \mu^{\tau_{\Lambda^c}}_{\beta,\gamma}(\tau_j)| \\ &\quad = \left| \int d\mu^{\tau_{\Lambda^c}}_{\beta,\gamma}(\tau_{\Lambda \setminus I}) \int d\mu^{\tau_{\Lambda^c}}_{\beta,\gamma}(\tau'_{\Lambda \setminus I}) \sum_{\tau_I, \tau'_I} Q^{\tau_{I^c}, \tau'_{I^c}}(\tau_I, \tau'_I) \, (\tau_i - \tau'_i) \, \tau_j \right| \\ &\quad \leqslant \int d\mu^{\tau_{\Lambda^c}}_{\beta,\gamma}(\tau_{\Lambda \setminus I}) \int d\mu^{\tau_{\Lambda^c}}_{\beta,\gamma}(\tau'_{\Lambda \setminus I}) \, D^{\tau_{I^c}, \tau'_{I^c}}_{I,i}. \end{aligned} \tag{4.12}$$

From Eqs. (4.10), (4.11), and (4.12) we conclude that there are constants $C_3, C_4 > 0$ such that, for any $\gamma \in (0, \gamma_\beta)$ and any boundary condition τ_{Λ^c},

$$|\mu^{\tau_{\Lambda^c}}_{\beta,\gamma}(\tau_i \tau_j)| \leqslant C_3 \left(e^{-C_4 \gamma |i-j|} + e^{-C_4 \gamma [\operatorname{dist}(i, \Lambda^c) + \operatorname{dist}(j, \Lambda^c)]} \right). \tag{4.13}$$

Let $\omega^2_{\mathcal{N}^b}$ be defined analogously to $\omega^2_{\mathcal{N}^b_k}$ in Eq. (4.4). By Eqs. (3.15) and (4.13) we have, for some constant $C_5 > 0$,

$$\nu^{\sigma_{\Delta_1 \cup \Delta_N}}_{\beta,\gamma} \left(\omega^2_{\mathcal{N}^b} \right) \leqslant 2 \sum_{i,j \in \mathcal{N}^b} \sup_{\tau_{\Delta_1 \cup \Delta_N}} \mu^{\tau_{\Delta_1 \cup \Delta_N}}_{\beta,\gamma}(\tau_i \tau_j) \leqslant N \frac{C_5}{\gamma}. \tag{4.14}$$

Then, recalling Eqs. (2.5), (2.3), (2.4), (3.6), and (3.12), the bound (2.7) easily follows from Eq. (4.14).

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